High order numerical schemes within a new first-order BSSNOK formulation of the Einstein equations

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Outline

1. First order formulations of the Einstein equations
2. Discretization of the Einstein equations
3. Numerical tests
4. The search for a new formulation
5. Conclusions

How does spacetime evolve?

Spacetime evolution

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \qquad ? \Longrightarrow \partial_t \mathbf{u} + \partial_i \mathbf{f}^i = \mathbf{S}$$

Fields evolution

- **continuity equation:**

  $\nabla_\alpha (\rho u^\alpha) = 0$

- **energy-momentum equation:**

  $\nabla_\mu \left(R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} \right) = 0 \Longrightarrow \nabla_\alpha T^{\alpha\beta} = 0, \qquad T^{\alpha\beta} = (e+p) u^\alpha u^\beta + p g^{\alpha\beta}$

- **electromagnetic fields evolution:**

  $\nabla_\alpha F^{*\alpha\beta} = 0, \qquad \nabla_\alpha F^{\alpha\beta} = 4\pi J^\beta \qquad \Longrightarrow \partial_t \mathbf{u} + \partial_i \mathbf{f}^i = \mathbf{S}$

Structure of the Einstein equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

- System of second-order non linear PDEs
- Second order in time derivatives exist only for the spatial components of $g_{\mu\nu}$, i.e. for g_{ij}.
- Einstein equations do not contain first order time derivatives of g_{tt} and $g_{ti} \Longrightarrow$ Freedom to introduce "gauge conditions".

Evolution sector

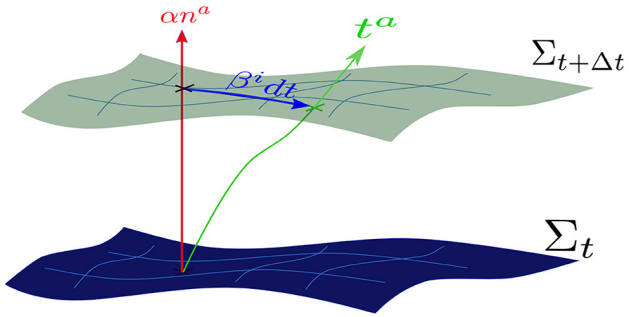
Six evolution equations for the spatial metric.

Constraint sector

Four constraints to be preserved in time:

$$R_{0\nu} - \frac{1}{2} R g_{0\nu} = \frac{8\pi G}{c^4} T_{0\nu}$$

Different spacetime approaches



$$ds^2 = -(\alpha^2 - \beta_i \beta^i) dt^2 + 2\beta_i \, dx^i \, dt + \gamma_{ij} dx^i \, dx^j \, ,$$

Generalized Harmonic formulation

- Evolve the full $g_{\mu\nu}$
- Naturally first order after placing:

$$\Phi_{i\mu\nu} = \partial_i g_{\mu\nu}$$
$$\Pi_{\mu\nu} = -n^\kappa \partial_\kappa g_{\mu\nu} \, .$$

3+1 formulations

- Evolve the spatial part γ_{ij}
- BSSNOK formulation (second order in space derivatives)
- Z4 formulation (specific for treating the Einstein constraints

$$G_{\mu\nu} + \nabla_\mu z_\nu + \nabla_\nu z_\mu - \nabla_\pi z^\pi g_{\mu\nu} = 8\pi T_{\mu\nu} \, ,$$

Most popular Numerical Relativity codes

Code	**Formulation**	**Scheme**	**Web Site**
Cactus	BSSNOK-SO	FD	http://einsteintoolkit.org
LazEv	BSSNOK-SO	FD	https://ccrg.rit.edu/content/software/lazev
BAM	BSSNOK-SO	FD	http://data.cardiffgravity.org/bam-catalogue/
GRChombo	BSSNOK-SO	FD	https://www.grchombo.org
SENR/NRPy	BSSNOK-SO	FD	https://www.black-holes.org/code/SpEC.html
SACRA	BSSNOK/Z4c-SO	FD	https://www.aei.mpg.de/1095648/SACRA-description
MHDueT	CCZ4-SO	FD	http://mhduet.liu.edu/
AMReX	CCZ4/Z4c-SO	FD	https://amrex-codes.github.io/amrex
Nmesh	GHG	DG	
SpEC	GHG	DG	https://www.black-holes.org/code/SpEC.html
ExaGRyPE	CCZ4	FV	https://hpcsoftware.pages.gitlab.lrz.de/Peano
CoCoNUT	CFC	spectral	https://www.uv.es/coconut/

The BSSNOK formulation as a second order system

- Key feature: The spatial metric is factorized conformally as:

$$\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij}$$

$$
\begin{aligned}
(\partial_t - \mathcal{L}_\beta)\, \tilde{\gamma}_{ij} &= -2\,\alpha\, \tilde{A}_{ij}\,, \\
(\partial_t - \mathcal{L}_\beta)\, \phi &= -\frac{1}{6}\alpha\, K\,, \\
(\partial_t - \mathcal{L}_\beta)\, \tilde{A}_{ij} &= e^{-4\phi}\left[-\nabla_i\nabla_j\alpha + \alpha\, R_{ij} - 8\pi\alpha S_{ij} \right]^{TF} + \alpha\left(K\,\tilde{A}_{ij} - 2\tilde{A}_{im}\tilde{A}^m_j\right)\,, \\
(\partial_t - \mathcal{L}_\beta)\, K &= -\nabla_i\nabla^i\alpha + \alpha\left(\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2 + 4\pi(E+S)\right)\,, \\
(\partial_t - \mathcal{L}_\beta)\, \tilde{\Gamma}^i &= \tilde{\gamma}^{jk}\partial_j\partial_k\beta^i + \frac{1}{3}\tilde{\gamma}^{ij}\partial_j\partial_k\beta^k - 2\tilde{A}^{ij}\partial_j\alpha \\
&+ 2\alpha\left(\tilde{\Gamma}^i_{jk}\tilde{A}^{jk} + 6\tilde{A}^{ij}\partial_j\phi - \frac{2}{3}\tilde{\gamma}^{ij}\partial_j K - 8\pi e^{4\phi} M^i\right)
\end{aligned}
$$

First order formulations explored in Trento...

	Ad hoc treatment of Einstein constraints	**Conformal factor**	**Key features**
Z4	YES	NO	Einstein constraints controlled by Z^μ
GH	YES	NO	Specific choice for the gauge conditions
BSSNOK	NO	YES	1) Use Hamiltonian constraint to replace R in the equation for $\partial_t K$ 2) The gamma-driver variables are evolved quantities
CCZ4	YES	YES	Combine BSSNOK and Z4

The BSSNOK formulation as a first order system

Suppose we introduce auxiliary variables [2, 1]

$$A_i := \partial_i \ln\alpha = \frac{\partial_i \alpha}{\alpha}\,, \qquad B_k{}^i := \partial_k \beta^i\,, \qquad D_{kij} := \frac{1}{2}\partial_k \gamma_{ij}\,.$$

$$\partial_t \tilde\gamma_{ij} - \beta^k \partial_k \tilde\gamma_{ij} = \tilde\gamma_{ik} B_j{}^k + \tilde\gamma_{kj} B_i{}^k - \frac{2}{3}\tilde\gamma_{ij} B_k{}^k - 2\alpha \tilde A_{ij}\,,$$

$$\partial_t \phi - \beta^k \partial_k \phi = \frac{1}{6} B_k{}^k - \frac{1}{6}\alpha K\,,$$

$$\partial_t \tilde A_{ij} - \beta^k \partial_k \tilde A_{ij} + \alpha e^{-4\phi}\left(\partial_{(i} A_{j)} - \frac{1}{3}\tilde\gamma_{ij}\tilde\gamma^{mn}\partial_{(m} A_{n)}\right) - \alpha e^{-4\phi}\left[(R_{ij})^{TF}_{ncp}\right] =$$

$$\tilde A_{ik} B_j{}^k + \tilde A_{kj} B_i{}^k - \frac{2}{3}\tilde A_{ij} B_k{}^k - \alpha e^{-4\phi}\left[A_i A_j - \Gamma^k_{ij} A_k - \frac{1}{3}\tilde\gamma_{ij}\tilde\gamma^{mn}(A_m A_n - \Gamma^k_{mn} A_k)\right] +$$

$$+ \alpha e^{-4\phi}\left[(R_{ij})^{TF}_{src}\right] - 8\pi\alpha e^{-4\phi}\left(S_{ij} - \frac{1}{3} e^{4\phi}\tilde\gamma_{ij}\, S\right) + \alpha(K\tilde A_{ij} - 2\tilde A_{il}\tilde\gamma^{lm}\tilde A_{mj})\,,$$

$$\partial_t K - \beta^k \partial_k K + \alpha e^{-4\phi}\tilde\gamma^{ij}\partial_{(i} A_{j)} = -\alpha e^{-4\phi}\tilde\gamma^{ij}\left(A_i A_j - \Gamma^k_{ij} A_k\right) + \alpha\left(\tilde A_{ij}\tilde A^{ij} + \frac{1}{3}K^2 + 4\pi(E+S)\right)$$

$$\partial_t \ln\alpha - \beta^k \partial_k \ln\alpha = -g(\alpha)\alpha(K - K_0)\,,$$

$$\partial_t \beta^i - b\, s\, \beta^k \partial_k \beta^i = \frac{3}{4}\, s\, b^i,$$

The BSSNOK formulation as a first order system

$$\partial_t b^i - b\, s\,(\beta^k \partial_k b^i - \beta^k \partial_k \tilde{\Gamma}^i) = s(\partial_t \tilde{\Gamma}^i - \eta b^i)\,,$$

$$\partial_t \tilde{\Gamma}^i - s\left[\beta^k \partial_k \tilde{\Gamma}^i + \tilde{\gamma}^{jk}\partial_{(j}B_{k)}^{\ \ i} + \frac{1}{3}\tilde{\gamma}^{ij}\partial_{(j}B_{k)}^{\ \ k} - \frac{4}{3}\alpha\tilde{\gamma}^{ij}\partial_j K\right] =$$

$$s\left[\frac{2}{3}\tilde{\Gamma}^i B_k^{\ k} - \tilde{\Gamma}^k B_k^{\ i} - 2\alpha\tilde{A}^{ij}A_j + 2\alpha\left(\tilde{\Gamma}^i_{jk}\tilde{A}^{jk} + 6\tilde{A}^{ij}P_j - 8\pi e^{4\phi}M^i\right)\right],$$

$$\partial_t A_i - \beta^k\partial_k A_i + \alpha g(\alpha)\,(\partial_i K - \partial_i K_0) = -\alpha A_i\,(K - K_0)\left(g(\alpha) + \alpha g'(\alpha)\right) + B_i^{\ k}\,A_k$$

$$\partial_t B_k^{\ i} - s\left[\frac{3}{4}\partial_k b^i + b\left(\beta^m\partial_m B_k^{\ i}\right) - \mu\,\alpha^2\gamma^{ij}\gamma^{nl}\left(\partial_k D_{ljn} - \partial_l D_{kjn}\right)\right] = s\,b\,B_m^{\ i}B_k^{\ m}\,,$$

$$\partial_t D_{kij} - \beta^m\partial_m D_{kij} - \frac{1}{2}\tilde{\gamma}_{mi}\partial_{(k}B_{j)}^{\ \ m} - \frac{1}{2}\tilde{\gamma}_{mj}\partial_{(k}B_{i)}^{\ \ m} + \frac{1}{3}\tilde{\gamma}_{ij}\partial_{(k}B_{m)}^{\ \ m} + \alpha\partial_k\tilde{A}_{ij} = B_k^{\ m}D_{mij} + B_j^{\ m}D_{kmi} + B_i^{\ m}D_{kmj}$$

$$-\frac{2}{3}B_m^{\ m}D_{kij} - \alpha A_k\tilde{A}_{ij} + \frac{1}{3}\alpha\tilde{\gamma}_{ij}\left[\tilde{\gamma}^{nm}\partial_k\tilde{A}_{nm} + \tilde{A}_{nm}\partial_k\tilde{\gamma}^{nm}\right],$$

$$\partial_t P_i - \beta^k\partial_k P_i + \frac{1}{6}\alpha\partial_i K - \frac{1}{6}\partial_{(i}B_{k)}^{\ \ k} = P_k B_i^{\ k} - \frac{1}{6}\alpha K A_i$$

The resulting system: 63 equations (Einstein + hydro)

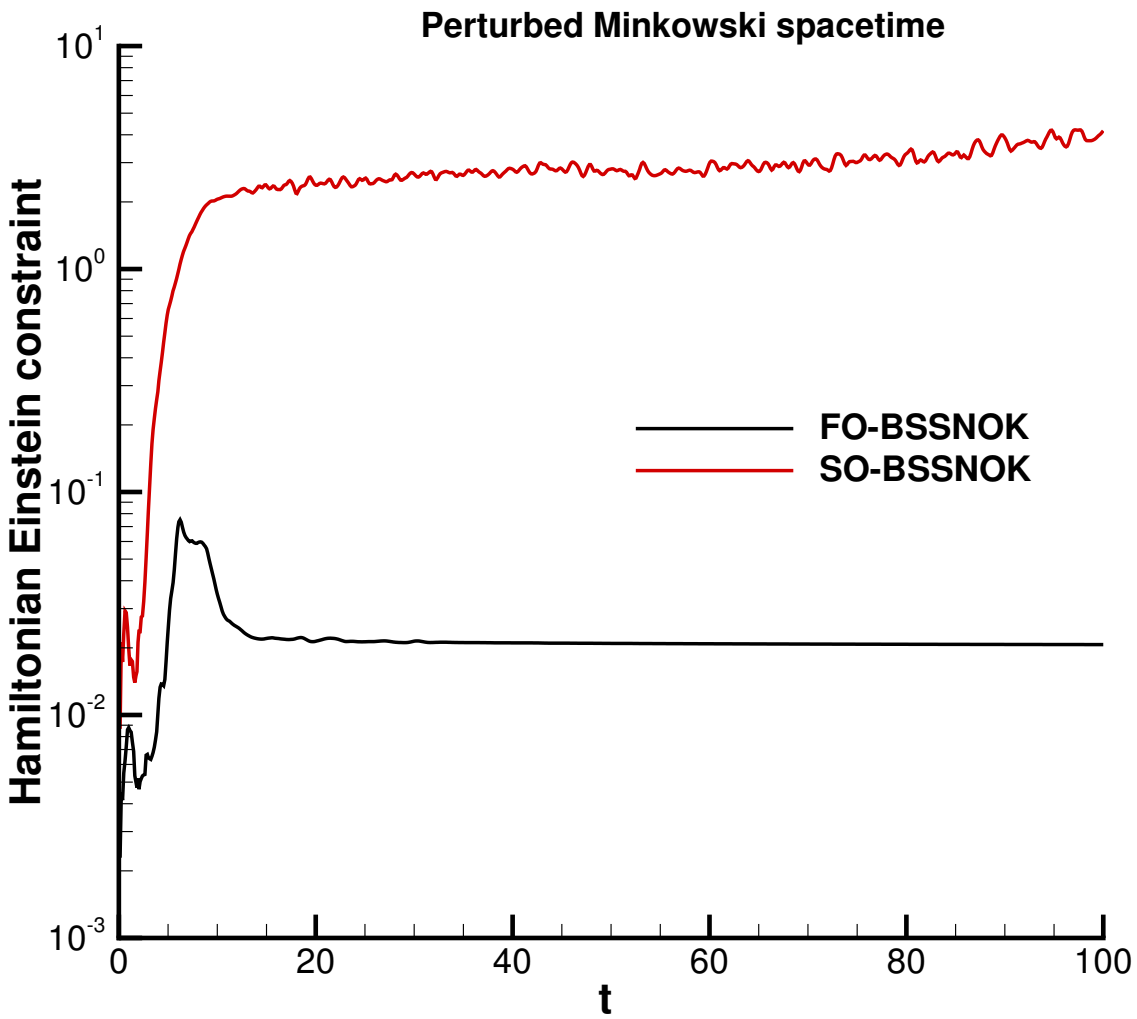
$$\frac{\partial \mathbf{u}}{\partial t} + \nabla\cdot\mathbf{F}(\mathbf{u}) + \mathbf{B}(\mathbf{u})\cdot\nabla\mathbf{u} = \mathbf{S}(\mathbf{u}),$$

Hyperbolicity

Major findings

- The new FO-BSSNOK system has real eigenvalues and a complete set of eigenvectors!
- DG schemes do not tolerate weak hyperbolicity
- WENO schemes can tolerate weak hyperbolicity

Comparing FO and SO BSSNOK



Numerical schemes adopted

	Numerical scheme
Z4/CCZ4	ADER-DG
GH	ADER-DG (unstructured polygonal meshes) FD-CWENO + Runge Kutta
FO-BSSNOK	ADER-DG (Cartesian meshes) FD-CWENO + Runge-Kutta

ADER DG scheme for non conservative systems

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \mathbf{F}(\mathbf{u}) + \mathbf{B}(\mathbf{u}) \cdot \nabla \mathbf{u} = \mathbf{S}(\mathbf{u}),$$

- The weak formulation of the PDE

$$\int_{t^n}^{t^{n+1}} \int_{\Omega_i} \Phi_k \, \frac{\partial \mathbf{u}}{\partial t} \, d\mathbf{x} \, dt + \int_{t^n}^{t^{n+1}} \int_{\Omega_i} \Phi_k \, \left(\nabla \cdot \mathbf{F}(\mathbf{u}) + \mathbf{B}(\mathbf{u}) \cdot \nabla \mathbf{u} \right) \, d\mathbf{x} \, dt = \int_{t^n}^{t^{n+1}} \int_{\Omega_i} \Phi_k \, \mathbf{S}(\mathbf{u}) \, d\mathbf{x} \, dt \, .$$

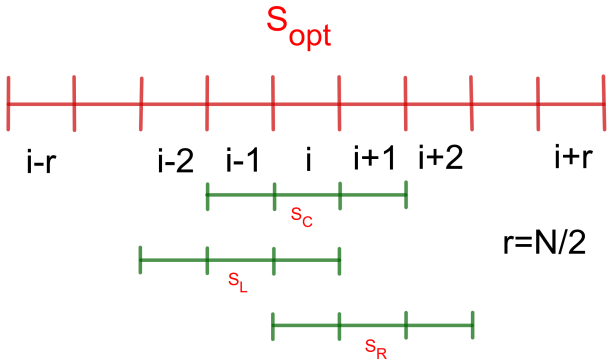
$$\mathbf{u}_h(\boldsymbol{x}, t^n) = \sum_{\ell} \boldsymbol{u}_{i,\ell}^n \Phi_\ell(\boldsymbol{x}) := \boldsymbol{u}_{i,\ell}^n \Phi_\ell(\boldsymbol{x}) \, .$$

- Use of the predictor
- Path conservative schemes for non the NC terms
- Sub-cell FV limiter
- Well balancing

FD conservative schemes in brief

- High order spatial reconstruction through CWENO schemes

$$\partial_x \mathbf{u}_i = \frac{\mathbf{w}^-_{i+\frac{1}{2}} - \mathbf{w}^+_{i-\frac{1}{2}}}{\Delta x},$$



$$P^N_{opt}(x) = \lambda_0 P^N_0(x) + \lambda_L P^2_L(x) + \lambda_C P^2_C(x) + \lambda_R P^2_R(x)\,.$$

- Semi-discrete (Runge-Kutta in time) discretization
- Path conservative schemes for non the NC terms
- Well Balancing

Runge–Kutta FD scheme for non conservative systems

$$\frac{d\mathbf{u}_{i,j,k}}{dt} = -\frac{\mathbf{f}^x_{i+\frac{1}{2},j,k} - \mathbf{f}^x_{i-\frac{1}{2},j,k}}{\Delta x} - \frac{\mathbf{f}^y_{i,j+\frac{1}{2},k} - \mathbf{f}^y_{i,j-\frac{1}{2},k}}{\Delta y} - \frac{\mathbf{f}^z_{i,j,k+\frac{1}{2}} - \mathbf{f}^z_{i,j,k-\frac{1}{2}}}{\Delta z} - \mathbf{B}(\mathbf{u}_{i,j,k}) \cdot \nabla \mathbf{u}_{i,j,k} + - \frac{\mathbf{D}^x_{i+\frac{1}{2},j,k} + \mathbf{D}^x_{i-\frac{1}{2},j,k}}{\Delta x} - \frac{\mathbf{D}^y_{i,j+\frac{1}{2},k} + \mathbf{D}^y_{i,j-\frac{1}{2},k}}{\Delta y} - \frac{\mathbf{D}^z_{i,j,k+\frac{1}{2}} + \mathbf{D}^z_{i,j,k-\frac{1}{2}}}{\Delta z} + \mathbf{S}(\mathbf{u}_{i,j,k}).$$

- Rusanov–type (local Lax–Friedrichs) dissipation:

$$\mathbf{D}^x_{i+\frac{1}{2},j,k} = \frac{1}{2}\tilde{\mathbf{B}}^x_{i+\frac{1}{2},j,k} \cdot \left(\mathbf{w}^+_{i+\frac{1}{2},j,k} - \mathbf{w}^-_{i+\frac{1}{2},j,k} \right),$$

- HLL–type dissipation:

$$\mathbf{D}^x_{i+\frac{1}{2},j,k} = -\frac{\lambda^-_{i+\frac{1}{2}}}{\lambda^+_{i+\frac{1}{2}} - \lambda^-_{i+\frac{1}{2}}} \tilde{\mathbf{B}}^x_{i+\frac{1}{2},j,k} \cdot \left(\mathbf{w}^+_{i+\frac{1}{2},j,k} - \mathbf{w}^-_{i+\frac{1}{2},j,k} \right),$$

$$\mathbf{D}^x_{i-\frac{1}{2},j,k} = \frac{\lambda^+_{i-\frac{1}{2}}}{\lambda^+_{i-\frac{1}{2}} - \lambda^-_{i-\frac{1}{2}}} \tilde{\mathbf{B}}^x_{i-\frac{1}{2},j,k} \cdot \left(\mathbf{w}^+_{i-\frac{1}{2},j,k} - \mathbf{w}^-_{i-\frac{1}{2},j,k} \right).$$

Gauge wave: GH formulation with DG

$$ds^2 = -H(x,t)\,dt^2 + H(x,t)\,dx^2 + dy^2 + dz^2,$$

where

$$H(x,t) = 1 - A\,\sin\left(2\pi(x-t)\right),$$

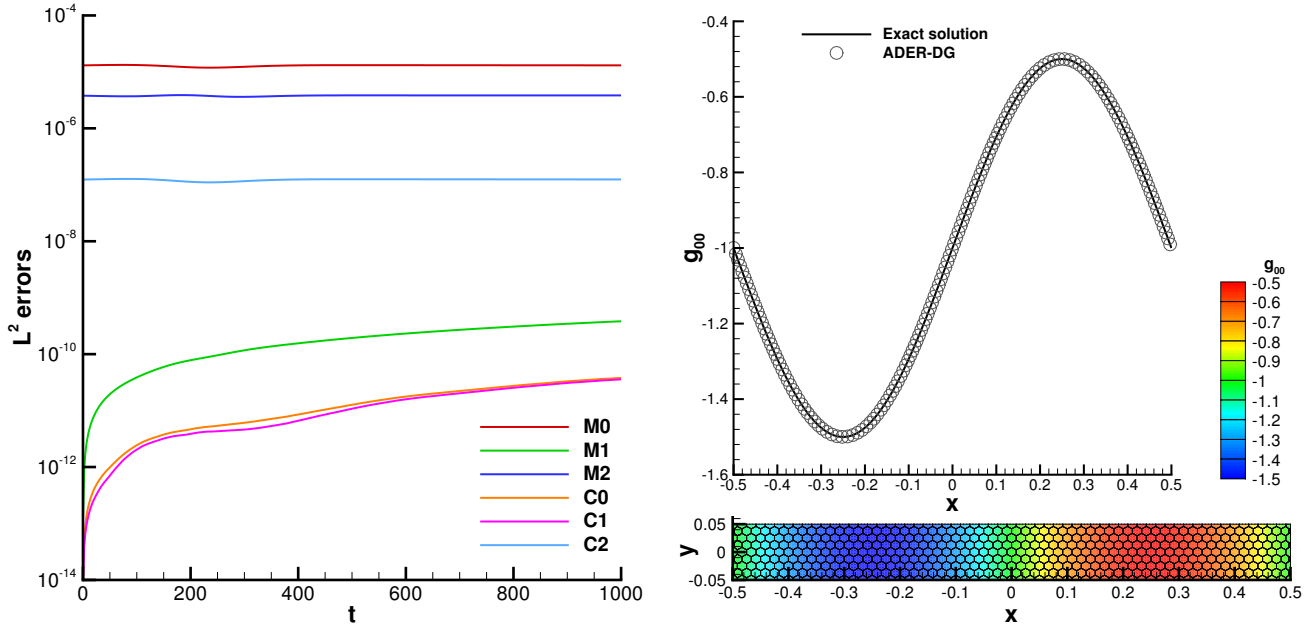
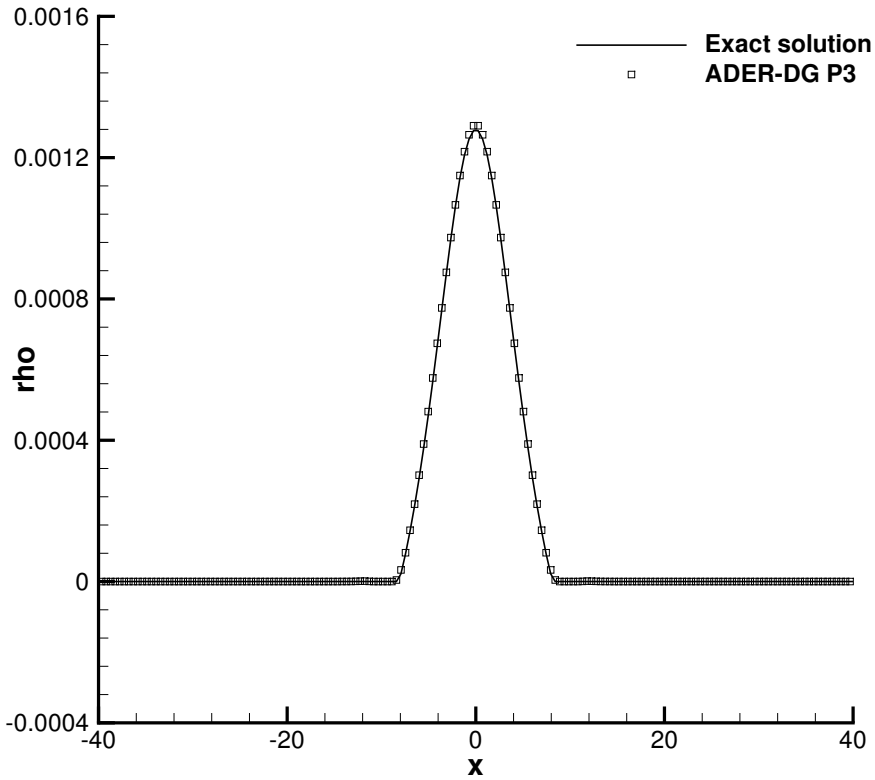


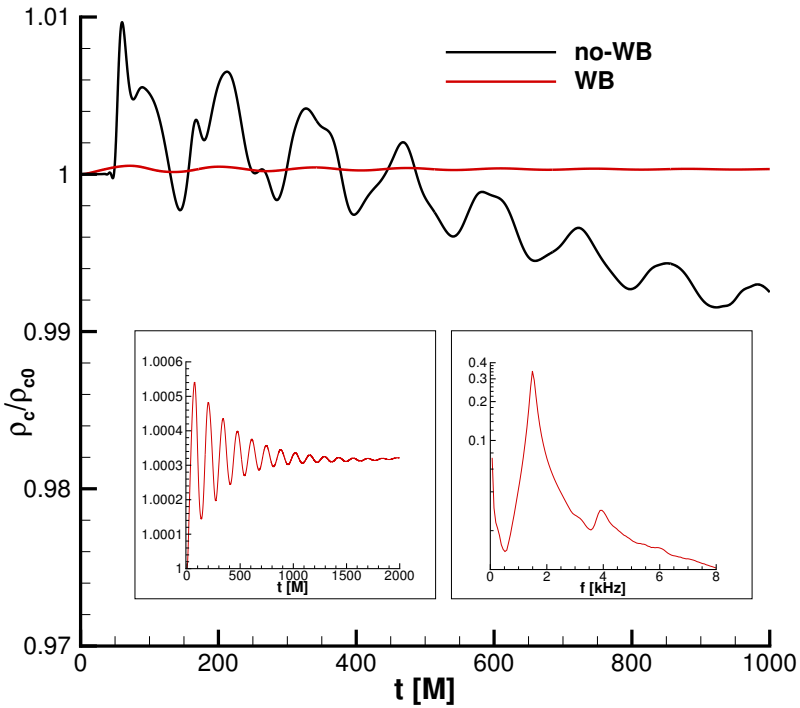
Figure: GH formulation

Numerical convergence: GH formulation with DG (Gauge wave)

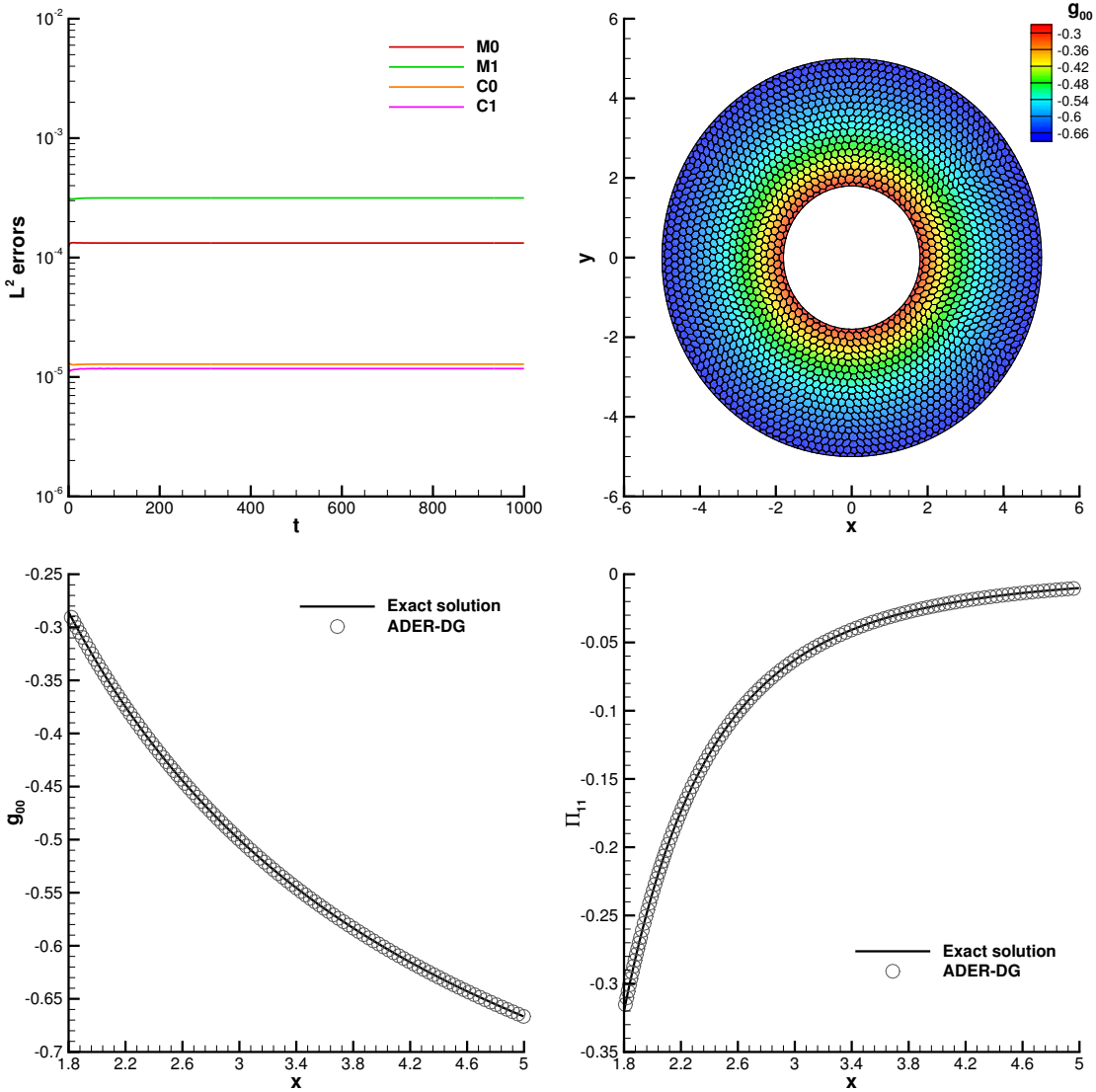
Scheme	N_x	h	L^2 Error g_{00}	$\mathcal{O}(g_{00})$	Theoretical
ADER-DG $\mathbb{P}_1$	40	8.3140E-03	1.7907E-04	-	
	80	4.4844E-03	4.4628E-05	2.25	2.00
	120	3.0898E-03	2.0248E-05	2.12	
ADER-DG $\mathbb{P}_2$	40	8.3140E-03	2.0386E-06	-	
	80	4.4844E-03	2.6027E-07	3.33	3.00
	120	3.0898E-03	7.7873E-08	3.24	
ADER-DG $\mathbb{P}_3$	40	8.3140E-03	1.5630E-08	-	
	80	4.4844E-03	1.0416E-09	4.39	4.00
	120	3.0898E-03	2.0965E-10	4.30	
ADER-DG $\mathbb{P}_4$	20	2.0127E-02	5.5749E-09	-	
	40	8.3140E-03	1.7908E-10	3.89	5.00
	80	4.4844E-03	9.9518E-12	4.68	

TOV star: FO-BSSNOK with DG/CWENO

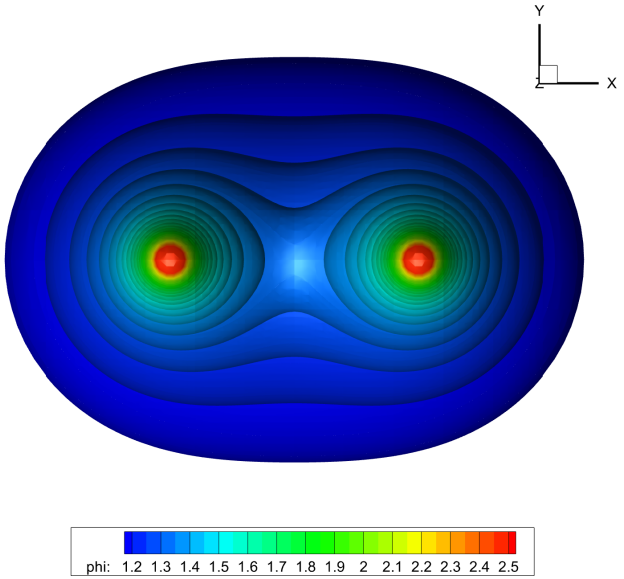


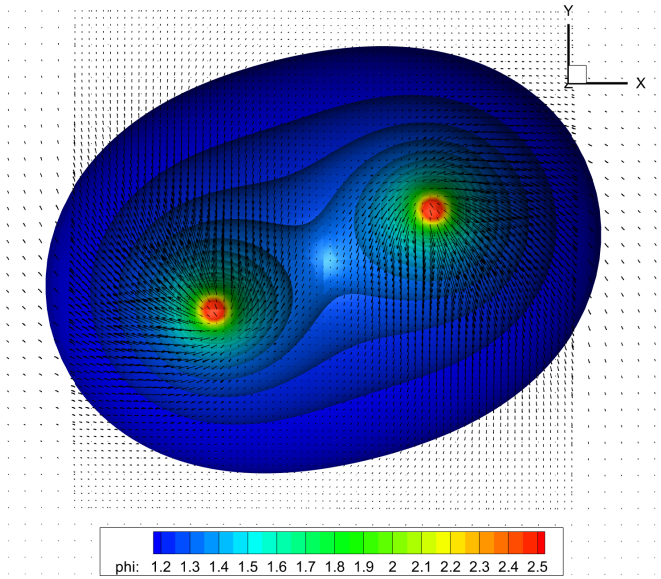


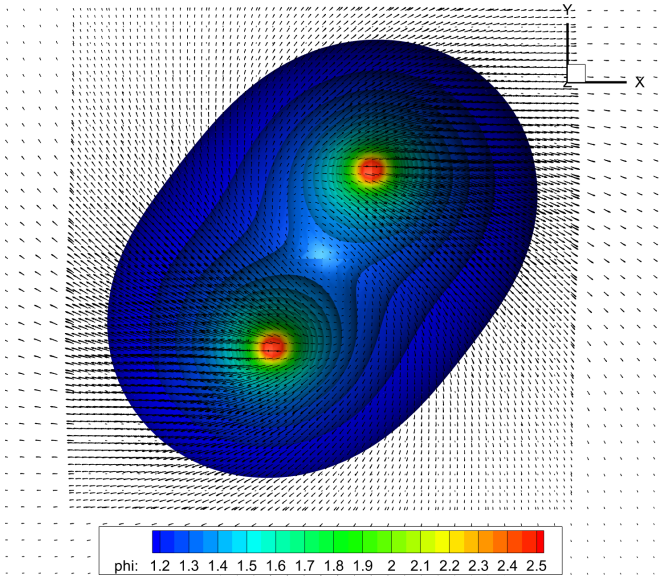
2D Schwarzschild BH with GH (fourth order ADER-DG)

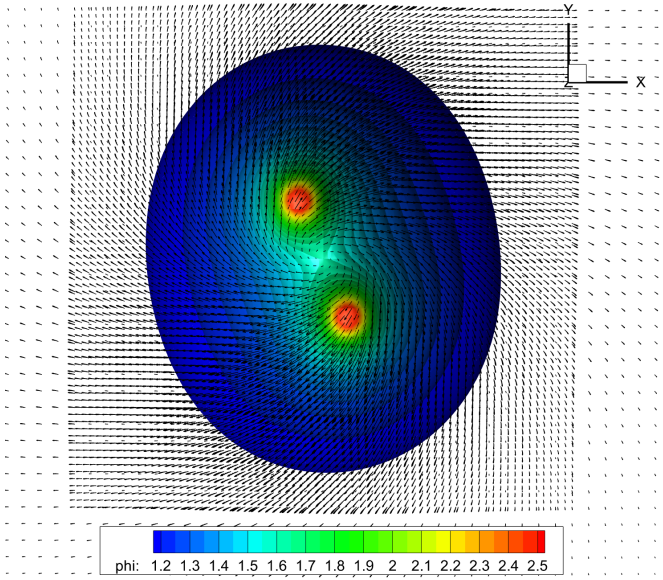


Numerical tests: Inspiralling binary merger with DG

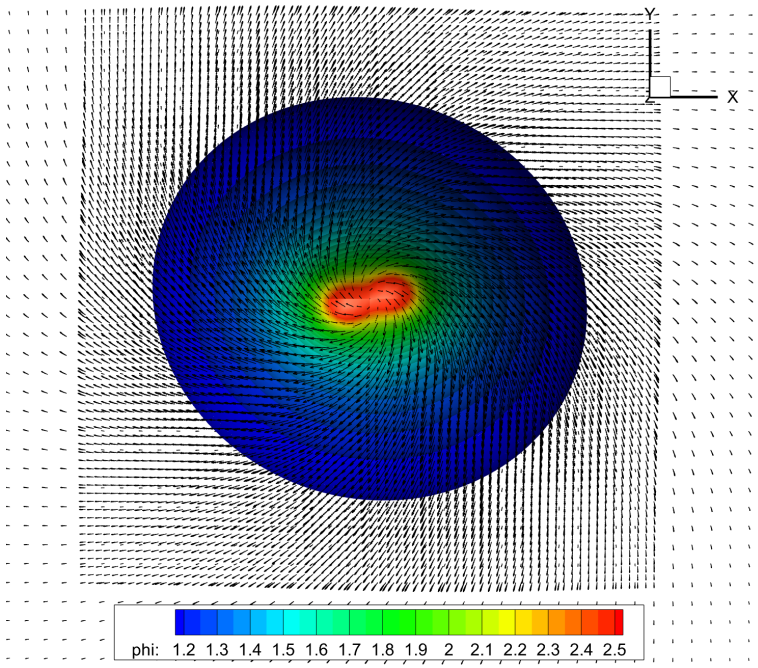


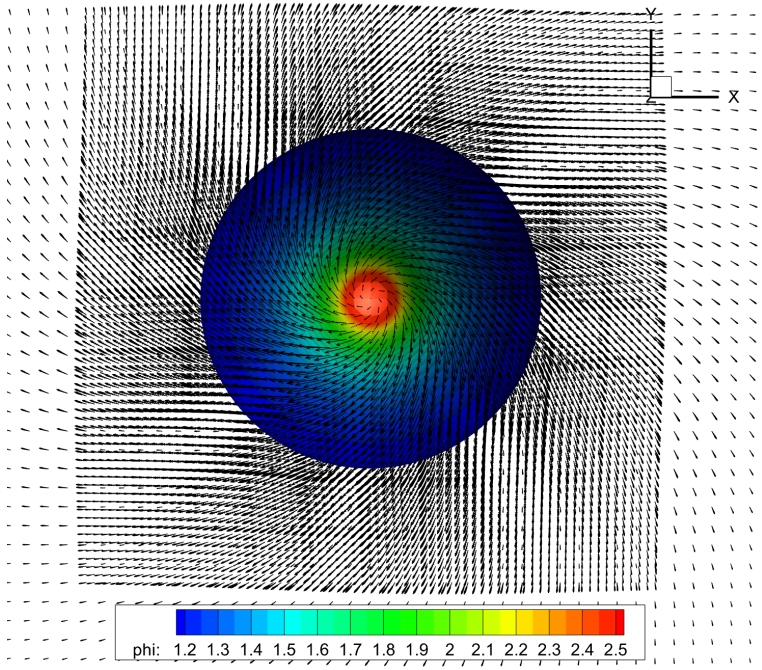




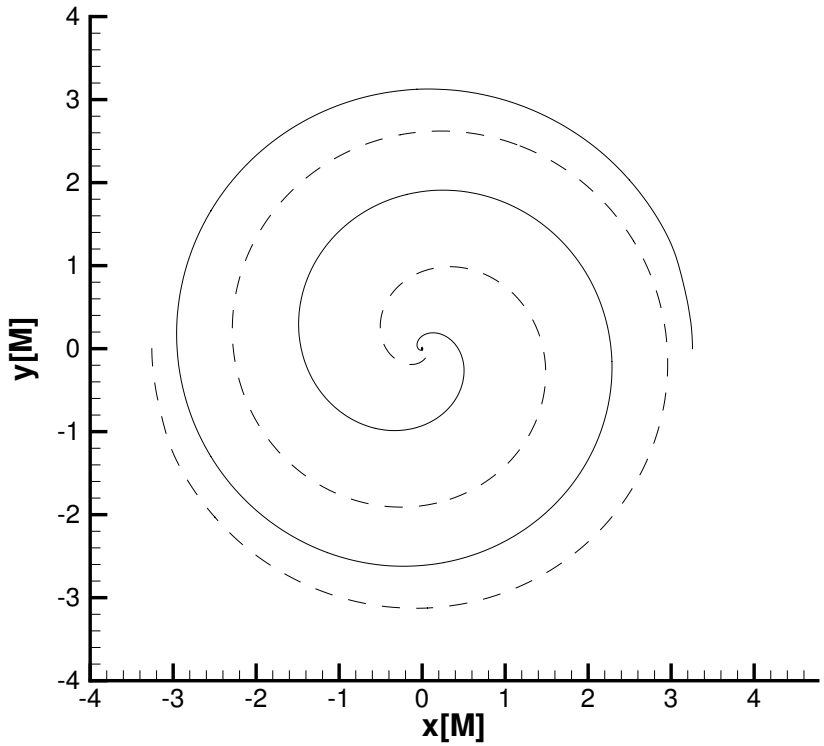


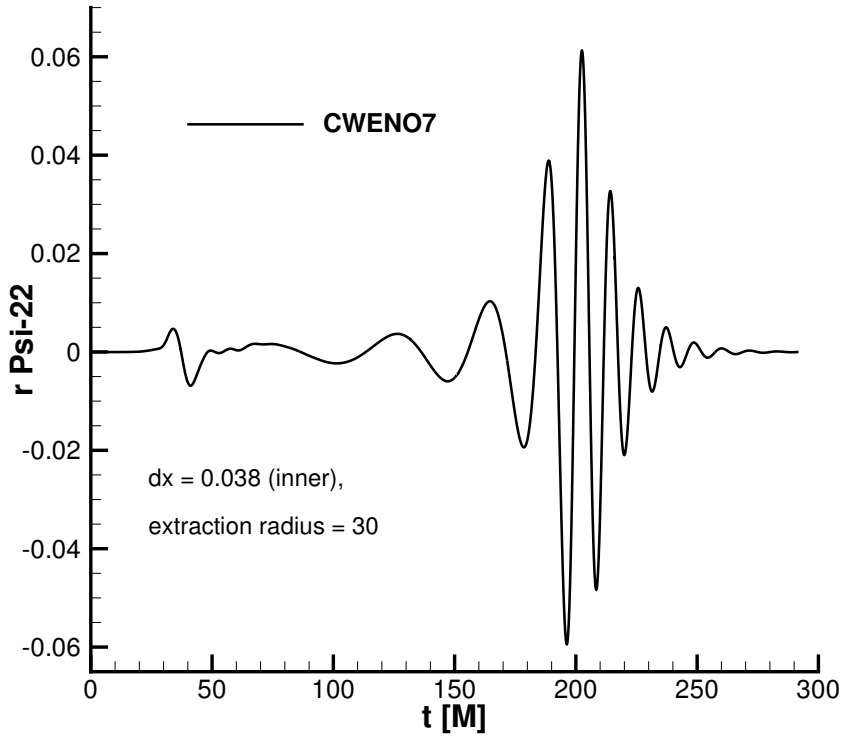
Numerical tests: Inspiralling binary merger with DG





Inspiralling BH binary merger: FO-BSSN with FD -CWENO





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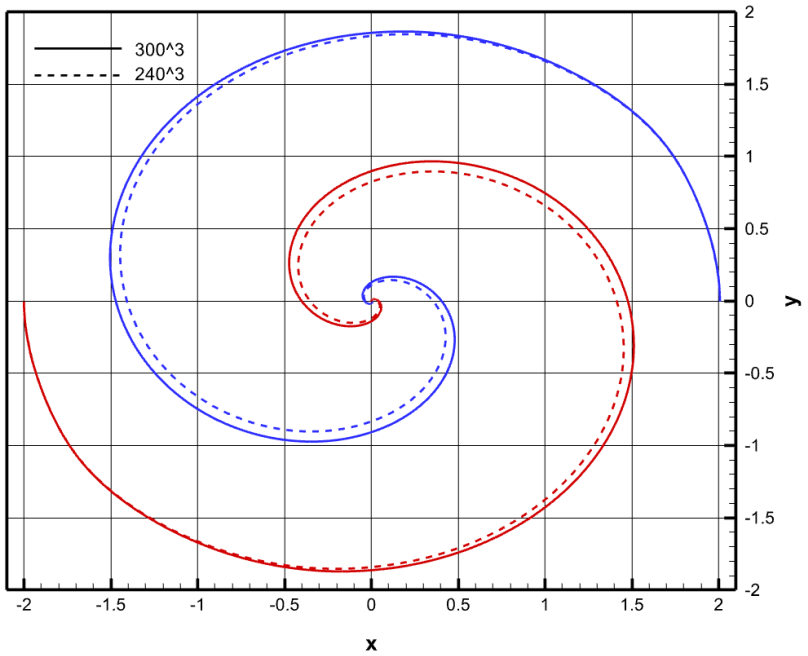
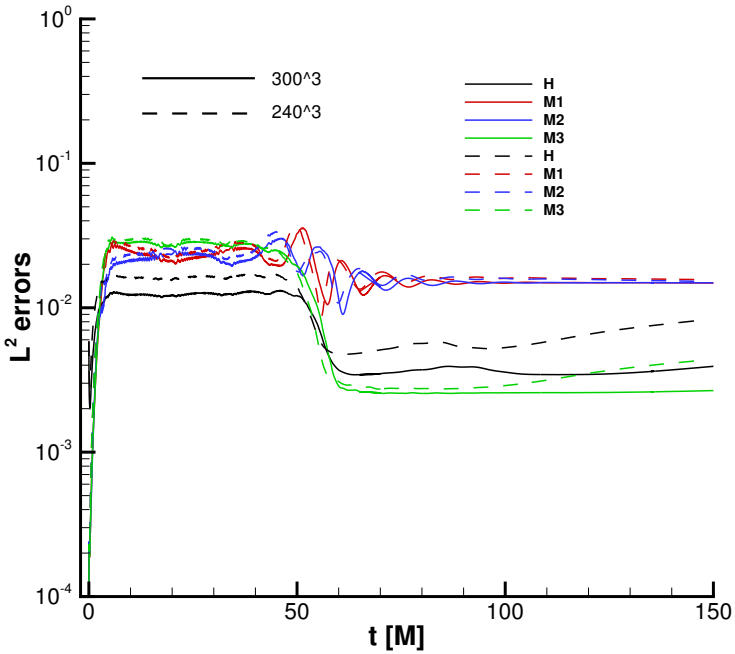
Numerical tests: Inspiralling binary merger with DG





CPU time/accuracy comparison among FD-CWENO and DG

2D test with the Gauge wave: the number of total elements is kept constant.

	Elements $= 20^3 = 8000$	
WCT (normalized)	FD-CWENO	ADER-DG
Order 3	1	81.81
Order 5	1	723.32
TEU (normalized)	FD-CWENO	ADER-DG
Order 3	1	14.02
Order 5	1	50.29
TDU (normalized)	FD-CWENO	ADER-DG
Order 3	1	0.52
Order 5	1	0.40
L2-error (lapse)	FD-CWENO	ADER-DG
Order 3	1.64E-03	6.05E-06
Order 5	2.43E-05	1.70E-09

CPU time/accuracy comparison among FD-CWENO and DG

2D test with the Gauge wave: the number of total degree of freedom is kept constant.

	DoFs = 216000	
WCT (normalized)	FD-CWENO	ADER-DG
Order 3	1	0.87
Order 5	1	1.09
TEU (normalized)	FD-CWENO	ADER-DG
Order 3	1	11.89
Order 5	1	46.50
TDU (normalized)	FD-CWENO	ADER-DG
Order 3	1	0.44
Order 5	1	0.37
L2-error (lapse)	FD-CWENO	ADER-DG
Order 3	4.97E-05	6.05E-06
Order 5	1.08E-07	2.09E-08

And yet...none of these formulations is perfect

	Gauge Wave	Schwarzschild BH	Binary BHs	Binary NSs
Z4	YES	YES	NO	NO
GH	YES	YES	YES,but complex	YES
BSSNOK	NO	NO	YES	YES

Conclusions

- We have been exploring various formulations of the Einstein equations, resulting in a new First Order version of the BSSNOK
- The new formulation can be combined either with Runge–Kutta FD or with DG schemes
- We have computed the first GW from the binary merger of two BHs obtained within a FO-BSSNOK formulation
- We are still looking for an optimal formulation...

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